

THE ξ -STABILITY ON THE AFFINE GRASSMANNIAN

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ABSTRACT. We introduce a notion of ξ -stability on the affine grassmannian \mathcal{X} for the classical groups. For the group SL_d , we calculate the Poincaré series of the quotient \mathcal{X}^ξ/T of the stable part \mathcal{X}^ξ by the maximal torus T by a process analogue to the Harder-Narasimhan reduction.

INTRODUCTION

Let k be an algebraically closed field, $F = k((\epsilon))$ the field of Laurent series with coefficients in k , $\mathcal{O} = k[[\epsilon]]$ the ring of integers of F , $\mathfrak{p} = \epsilon k[[\epsilon]]$ the maximal ideal of \mathcal{O} . Let $\mathrm{val} : F^\times \rightarrow \mathbf{Z}$ be the discrete valuation normalized by $\mathrm{val}(\epsilon) = 1$.

Let G be a classical group over k , let T be a maximal torus of G . Let $K = G(\mathcal{O})$ be the standard maximal compact subgroup of $G(F)$. Let $\mathcal{X}^G = G(F)/K$ be the affine grassmannian associated to G . We simplify \mathcal{X}^G to \mathcal{X} when the context is clear. We introduce a notion of ξ -stability on the affine grassmannian \mathcal{X} , which is a local version of the ξ -stability on the Hitchin space introduced by Chaudouard and Laumon. First of all, we show that the quotient \mathcal{X}^ξ/T of the stable part \mathcal{X}^ξ by the torus T exists as an $\mathrm{ind}\text{-}k$ -scheme, by a calculation with the geometric invariant theory of Mumford. Then we introduce a reduction process which permits to reduce the non- ξ -stable parts onto the ξ^M -stable parts on the affine grassmannian associated to the Levi subgroups of G containing T . Finally, for the group SL_d , we calculate the Poincaré series of \mathcal{X}^ξ/T . There are two ingredients in the proof: the first one is the fact that \mathcal{X}^ξ/T is homologically smooth and hence satisfies the Poincaré duality, the second one is to give a lower bound on the codimension of the non- ξ -stable parts. The main result can be summarized in the following theorem.

Theorem 0.1. *Let G be a classical group over k , let T be a maximal torus of G . The geometric quotient \mathcal{X}^ξ/T exists as an $\mathrm{ind}\text{-}k$ -scheme, and it satisfies the valuative criterion of properness. For $G = \mathrm{SL}_d$, the Poincaré series of \mathcal{X}^ξ/T is*

$$\frac{1}{(1-t^2)^{d-1}} \prod_{i=1}^{d-1} (1-t^{2i})^{-1}.$$

Notations. Let $\Phi = \Phi(G, T)$ be the root system of G with respect to T , let W be the Weyl group of G with respect to T , and let \widetilde{W} be the extended affine Weyl group. For any subgroup H of G which is stable under the conjugation of T , we note $\Phi(H, T)$ for the roots appearing in $\mathrm{Lie}(H)$. We use the (G, M) notation of Arthur. Let $\mathcal{F}(T)$ be the

set of parabolic subgroups of G containing T , let $\mathcal{L}(T)$ be the set of Levi subgroups of G containing T . For every $M \in \mathcal{L}(T)$, we denote by $\mathcal{P}(M)$ the set of parabolic subgroups of G whose Levi factor is M . Let $X^*(M) = \text{Hom}(M, \mathbb{G}_m)$ and $\mathfrak{a}_M^* = X^*(M) \otimes \mathbf{R}$. The restriction $X^*(M) \rightarrow X^*(T)$ induces an injection $\mathfrak{a}_M^* \hookrightarrow \mathfrak{a}_T^*$. Let $(\mathfrak{a}_T^M)^*$ be the subspace of \mathfrak{a}_T^* generated by $\Phi(M, T)$. We have the decomposition in direct sums

$$\mathfrak{a}_T^* = (\mathfrak{a}_T^M)^* \oplus \mathfrak{a}_M^*.$$

The canonical pairing

$$X_*(T) \times X^*(T) \rightarrow \mathbf{Z}$$

can be extended linearly to $\mathfrak{a}_T \times \mathfrak{a}_T^* \rightarrow \mathbf{R}$, with $\mathfrak{a}_T = X_*(T) \otimes \mathbf{R}$. For $M \in \mathcal{L}(T)$, let $\mathfrak{a}_T^M \subset \mathfrak{a}_T$ be the subspace orthogonal to \mathfrak{a}_M^* , and $\mathfrak{a}_M \subset \mathfrak{a}_T$ be the subspace orthogonal to $(\mathfrak{a}_T^M)^*$, then we have the decomposition

$$\mathfrak{a}_T = \mathfrak{a}_M \oplus \mathfrak{a}_T^M,$$

let π_M, π^M be the projections to the two factors.

For $M \in \mathcal{L}(T)$, we use Λ_M to denote the quotient of $X_*(T)$ by the coroot lattice of M (the subgroup of $X_*(T)$ generated by the coroots of T in M). We have a canonical homomorphism

$$\text{ind}^M : M(F) \rightarrow \Lambda_M$$

such that $\chi(\text{ind}^M(m)) = \text{val}(\chi(m))$, $\forall \chi \in X^*(M)$. It is invariant under the right translation of $M(\mathcal{O})$, so it defines an application $\text{ind}^M : \mathcal{X}^M \rightarrow \Lambda_M$. Its fibers $\mathcal{X}^{M,(\lambda)} := \text{ind}^{-1}(\lambda)$, $\lambda \in \Lambda$ are the connected components of \mathcal{X}^M , they are all translations of the neutral connected component $\mathcal{X}^{M,(0)}$. For $M = \text{GL}_{d_1} \times \cdots \times \text{GL}_{d_r}$, Λ_M can be identified naturally with \mathbf{Z}^r , and the application $\text{ind}^M : \mathcal{X}^M \rightarrow \mathbf{Z}^r$ is nothing but

$$\text{ind}^M((m_1, \dots, m_r)) = (\text{val}(\det(m_1)), \dots, \text{val}(\det(m_r))).$$

For any point $x \in \mathcal{X}^M$, we call $\text{ind}^M(x)$ the index of the lattice represented by x . For $M = \text{GL}_d$, we simplify ind^{GL_d} to ind .

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1. THE IND- k -SCHEME \mathcal{X}^ξ/T

1.1. The notion of ξ -stabilty. For $M \in \mathcal{L}(T)$, the natural inclusion of $M(F)$ in $G(F)$ induces a closed immersion of \mathcal{X}^M in \mathcal{X}^G . For $P = MN \in \mathcal{F}(T)$, we have the retraction

$$f_P : \mathcal{X} \rightarrow \mathcal{X}^M$$

which sends $gK = nmK$ to $mM(\mathcal{O})$, where $g = nmk$, $n \in N(F)$, $m \in M(F)$, $k \in K$ is the Iwasawa decomposition. More generally we can define $f_{P_L}^L : \mathcal{X}^L \rightarrow \mathcal{X}^M$ for $L \in \mathcal{L}(T)$, $L \supset M$ and $P_L \in \mathcal{P}^L(M)$. These retractions satisfy the transition property: Suppose that $Q \in \mathcal{P}(L)$ satisfy $Q \supset P$, then

$$f_P = f_{P \cap L}^L \circ f_Q.$$

We have the function $H_P : \mathcal{X} \rightarrow \mathfrak{a}_M^G = \mathfrak{a}_M / \mathfrak{a}_G$ which is the composition of $\text{ind}^M \circ f_P$ and the natural projection of Λ_M to \mathfrak{a}_M^G .

Proposition 1.1 (Arthur). *Let $B', B'' \in \mathcal{P}(T)$ be two adjacent Borel subgroups, let $\alpha_{B', B''}^\vee$ be the coroot which is positive with respect to B' and negative with respect to B'' . Then for any $x \in \mathcal{X}$, we have*

$$H_{B'}(x) - H_{B''}(x) = n(x, B', B'') \cdot \alpha_{B', B''}^\vee,$$

with $n(x, B', B'') \in \mathbf{N}$.

Proof. Let P be the parabolic subgroup generated by B' and B'' , let $P = MN$ be the Levi factorization. The application $H_{B'}$ factor through f_P , i.e. we have commutative diagram

$$\begin{array}{ccc} \mathcal{X} & & \\ f_P \downarrow & \searrow H_{B'} & \\ \mathcal{X}^M & \xrightarrow{H_{B' \cap M}^M} & \mathfrak{a}_T^G \end{array}$$

and similarly for $H_{B''}$. Since M has semisimple rank 1, the proposition is thus reduced to

$G = \text{SL}_2$. In this case, let T be the maximal torus of the diagonal matrices, $B' = \begin{pmatrix} * & * \\ & * \end{pmatrix}$, $B'' = \begin{pmatrix} * & \\ * & * \end{pmatrix}$, and we identify \mathfrak{a}_T^G with the line $H = \{(x, -x) \mid x \in \mathbf{R}\} \subset \mathbf{R}^2$ in the usual

way. By the Iwasawa decomposition, any point $x \in \mathcal{X}$ can be written as $x = \begin{pmatrix} a & b \\ & d \end{pmatrix} K$. Let $m = \min\{\text{val}(a), \text{val}(b)\}$, $n = \text{val}(d)$, then $m + n \leq \text{val}(a) + \text{val}(d) = 0$ and

$$H_{B'}(x) = (-n, n), \quad H_{B''}(x) = (m, -m).$$

So

$$H_{B'}(x) - H_{B''}(x) = (-(n+m), n+m) = -(n+m) \cdot \alpha_{B', B''}^\vee,$$

and the proposition follows. \square

Definition 1.1. For any point $x \in \mathcal{X}$, we denote by $\text{Ec}(x)$ the convex envelope in \mathfrak{a}_T^G of the $H_{B'}(x)$, $B' \in \mathcal{P}(T)$.

Definition 1.2. Let $\xi \in \mathfrak{a}_T^G$, it is said to be generic if $\alpha(\xi) \notin \mathbf{Z}$, $\forall \alpha \in \Phi(G, T)$.

In the following, we always suppose that ξ is generic.

Definition 1.3. For any point $x \in \mathcal{X}$, we say that it is ξ -stable if $\xi \in \text{Ec}(x)$.

Let \mathcal{X}^ξ be the open sub-ind- k -scheme of \mathcal{X} of the ξ -stable points. It is invariant under the action of the maximal torus T . Since ξ is generic, the action of T/Z_G on \mathcal{X}^ξ is free, where Z_G is the center of G . For $\xi, \xi' \in \mathfrak{a}_T^G$, there exists $w \in \widehat{W}$ such that $\mathcal{X}^\xi = w\mathcal{X}^{\xi'}$. So the “quotient” \mathcal{X}^ξ/T is independent of the choice of ξ .

The aim of the rest of this section is to establish the following proposition.

Proposition 1.2. *The geometric quotient \mathcal{X}^ξ/T of \mathcal{X}^ξ by T exists as an ind- k -scheme. In fact, it is the direct limit of a family of projective varieties over k .*

Although the proof is a case by case analysis, the main idea rests the same. So we will give a detailed proof only for GL_d , and indicate the modifications for the other classical groups.

1.2. The group GL_d . Let $G = \mathrm{GL}_d$, let T be the maximal torus of the diagonal matrices. The affine grassmannian classifies the lattices in F^d , i.e.

$$\mathcal{X} = \{L \subset F^d \mid L \text{ is an } \mathcal{O}\text{-module of finite type such that } L \cdot F = F^d\}.$$

Since all the connected components of \mathcal{X} are translations of the neutral connected component $\mathcal{X}^{(0)}$, it is enough to study $\mathcal{X}^{(0)}$. Let $\{e_i\}_{i=1}^d$ be the natural basis of F^d over F .

Proposition 1.3. *Let $\xi \in \mathfrak{t}$ be such that $\sum_{i=1}^d \xi_i = 0$. A lattice $L \in \mathcal{X}$ of index 0 is ξ -stable if and only if for any permutation $\tau \in \mathfrak{S}_d$, we have*

$$\xi_{\tau(1)} + \cdots + \xi_{\tau(i)} \leq \mathrm{ind}(L \cap (Fe_{\tau(1)} \oplus \cdots \oplus Fe_{\tau(i)})), \quad i = 1, \dots, d.$$

Proof. Let $B' = \tau(B)$, let $H_{B'}(L) = (n_1, \dots, n_d)$, then we have

$$n_{\tau(1)} + \cdots + n_{\tau(i)} = \mathrm{ind}(L \cap (Fe_{\tau(1)} \oplus \cdots \oplus Fe_{\tau(i)})),$$

and the proposition follows. \square

Let $L_0 = \mathcal{O}^d$, let \mathcal{X}_n be the closed sub-scheme of $\mathcal{X}^{(0)}$ defined by

$$\mathcal{X}_n = \{L \in \mathcal{X}^{(0)} \mid L \supset \epsilon^n L_0\}.$$

It is a T -invariant projective k -variety, and we have

$$\mathcal{X}^{(0)} = \varinjlim_n \mathcal{X}_n.$$

We will prove the following result, which implies the proposition 1.2.

Proposition 1.4. *The quotient \mathcal{X}_n^ξ/T is a projective k -variety.*

1.2.1. A non-standard quotient of the grassmannian. Let E_1, \dots, E_d be vector spaces over k of dimension $\dim(E_i) = N_i$. Let X be the grassmannian of sub vector spaces of dimension n of $E_1 \oplus \cdots \oplus E_d$. We have the Plücker immersion $\iota : X \rightarrow \mathbf{P}^N$, $N = \binom{N_1 + \cdots + N_d}{n} - 1$, and the line bundle $\mathcal{L} = \iota^* \mathcal{O}_{\mathbf{P}^N}(1)$ is naturally endowed with a $\mathrm{Aut}(E_1 \oplus \cdots \oplus E_d)$ -linearization.

The torus $T = \mathbb{G}_m^d$ acts on $E_1 \oplus \cdots \oplus E_d$ with its i -th factor acts as homothetic on E_i , thus it acts on X . Given $r_i, s_i \in \mathbf{N}$, let $S \subset T$ be the sub torus of T defined by

$$(1) \quad S = \left\{ \begin{bmatrix} t_1^{r_1} & & & \\ & \ddots & & \\ & & t_{d-1}^{r_{d-1}} & \\ & & & t_1^{-s_1} \cdots t_{d-1}^{-s_{d-1}} \end{bmatrix} ; t_i \in k^\times \right\}.$$

We will give a geometric description of the (semi-)stable points of X under the action of S with respect to the polarization given by the line bundle \mathcal{L} , using the criteria of Hilbert-Mumford. Let Z be a projective algebraic variety over k endowed with the action of a reductive group H over k , let \mathcal{L} be an ample H -equivariant line bundle over Z . Let $\lambda : \mathbb{G}_m \rightarrow H$ be a homomorphism of algebraic group, then \mathbb{G}_m acts on Z via the morphism λ . For any point $x \in Z$, the point $x_0 := \lim_{t \rightarrow 0} \lambda(t)x$ exists since Z is projective. Obviously $x_0 \in Z^{\mathbb{G}_m}$, thus \mathbb{G}_m acts on the stalk \mathcal{L}_{x_0} . The action is given by a character of \mathbb{G}_m , $\alpha : t \rightarrow t^r$, for some $r \in \mathbf{Z}$. Let $\mu^{\mathcal{L}}(x, \lambda) = -r$.

Theorem 1.5 (Hilbert-Mumford). *Let Z be a projective algebraic variety over k endowed with the action of a reductive group H over k , let \mathcal{L} be an ample H -equivariant line bundle over Z . Let Z^{ss} (resp. Z^{st}) be the open sub variety of Z of the semi-stable (resp. stable) points. Then for any geometric point $x \in Z$, we have*

- (1) $x \in Z^{ss} \iff \mu^{\mathcal{L}}(x, \lambda) \geq 0, \forall \lambda \in \text{Hom}(\mathbb{G}_m, H),$
- (2) $x \in Z^{st} \iff \mu^{\mathcal{L}}(x, \lambda) > 0, \forall \lambda \in \text{Hom}(\mathbb{G}_m, H).$

For the proof, the reader can consult [M], page 49-54.

Lemma 1.6. *We have*

$$V \in X^S \iff V = V_1 \oplus \cdots \oplus V_d,$$

where $V_i \subset E_i$ is a sub vector space.

For $\mathbf{n} = (n_1, \dots, n_{d-1}) \in \mathbf{Z}^{d-1}$ such that the cocharacter $\lambda_{\mathbf{n}} \in X_*(S)$ defined by

$$\lambda_{\mathbf{n}}(t) = \begin{bmatrix} t^{n_1 r_1} & & & \\ & \ddots & & \\ & & t^{n_{d-1} r_{d-1}} & \\ & & & t^{-\sum_{i=1}^{d-1} n_i s_i} \end{bmatrix}$$

is regular. The stability condition is equivalent to the condition that $-\mu^{\mathcal{L}}(V, \lambda_{\mathbf{n}}) < 0$ for all such $\mathbf{n} \in \mathbf{Z}^{d-1}$. Up to conjugation, we can suppose that

$$(2) \quad n_1 r_1 < \cdots < n_i r_i < -\sum_{j=1}^{d-1} n_j s_j < n_{i+1} r_{i+1} < \cdots < n_{d-1} r_{d-1}.$$

Let $\lim_{t \rightarrow 0} \lambda_{\mathbf{n}}(t)V = V_1 \oplus \cdots \oplus V_d$, where $V_i \subset E_i$ is a sub vector space of dimension a_i . We have the relation

$$\begin{aligned}
(3) \quad a_1 &= \dim(V) - \dim(V \cap (E_2 \oplus \cdots \oplus E_d)), \\
&\vdots \\
a_i &= \dim(V \cap (E_i \oplus \cdots \oplus E_d)) - \dim(V \cap (E_{i+1} \oplus \cdots \oplus E_d)), \\
a_d &= \dim(V \cap (E_{i+1} \oplus \cdots \oplus E_d)) - \dim(V \cap (E_{i+1} \oplus \cdots \oplus E_{d-1})), \\
a_{i+1} &= \dim(V \cap (E_{i+1} \oplus \cdots \oplus E_{d-1})) - \dim(V \cap (E_{i+2} \oplus \cdots \oplus E_{d-1})), \\
&\vdots \\
a_{d-1} &= \dim(V \cap E_{d-1}).
\end{aligned}$$

So the stability condition can be written as

$$\begin{aligned}
(4) \quad -\mu^{\mathcal{L}}(V, \lambda_{\mathbf{n}}) &= \sum_{i=1}^{d-1} a_i n_i r_i - a_d \sum_{j=1}^{d-1} n_j s_j \\
&= \sum_{i=2}^{d-1} a_i (n_i r_i - n_1 r_1) - a_d \left(n_1 r_1 + \sum_{j=1}^{d-1} n_i s_i \right) + n n_1 r_1 < 0,
\end{aligned}$$

(We use the relation $n = a_1 + \cdots + a_d$ in the second equality.)

The equality (4) is a question of maximal value of a linear functional on a convex region, so it suffices to look at the condition at the boundary, i.e.

$$\begin{aligned}
(5) \quad n_1 r_1 = \cdots = n_{i_0} r_{i_0} < n_{i_0+1} r_{i_0+1} &= \cdots = - \sum_{j=1}^{d-1} n_j s_j \\
&= \cdots = n_{d-1} r_{d-1}, \quad 1 \leq i_0 \leq i;
\end{aligned}$$

and

$$\begin{aligned}
(6) \quad n_1 r_1 = \cdots = - \sum_{j=1}^{d-1} n_j s_j = \cdots &= n_{i_0} r_{i_0} < n_{i_0+1} r_{i_0+1} \\
&= \cdots = n_{d-1} r_{d-1}, \quad i+1 \leq i_0 \leq d-1.
\end{aligned}$$

The inequality (5) gives

$$n_1 < 0, \quad n_{i_0+1} r_{i_0+1} \left(1 + \sum_{j=i_0+1}^{d-1} s_j / r_j \right) = -n_1 r_1 \sum_{j=1}^{i_0} s_j / r_j,$$

so the inequality (4) implies

$$(7) \quad a_{i_0+1} + \cdots + a_d < \frac{n(1 + \sum_{j=i_0+1}^{d-1} s_j/r_j)}{1 + \sum_{j=1}^{d-1} s_j/r_j}, \quad 1 \leq i_0 \leq i.$$

The inequality (6) gives

$$n_1 < 0, \quad n_{i_0+1}r_{i_0+1} \sum_{j=i_0+1}^{d-1} s_j/r_j = -n_1r_1(1 + \sum_{j=1}^{i_0} s_j/r_j),$$

so the inequality (4) implies

$$(8) \quad a_{i_0+1} + \cdots + a_{d-1} < \frac{n(\sum_{j=i_0+1}^{d-1} s_j/r_j)}{1 + \sum_{j=1}^{d-1} s_j/r_j}, \quad i+1 \leq i_0 \leq d-1.$$

We can express the inequalities (7) and (8) as inequalities in $\dim(V \cap (E_{i_1} \oplus \cdots \oplus E_{i_r}))$ with the help of the dimension relation (3).

Let $x = (x_1, \dots, x_d) \in \mathfrak{t}$ with

$$x_d = \frac{1}{1 + \sum_{i=1}^{d-1} s_i/r_i}, \quad x_i = \frac{s_i/r_i}{1 + \sum_{i=1}^{d-1} s_i/r_i}, \quad i = 1, \dots, d-1.$$

We remark that $\sum_{i=1}^d x_i = 1$. The above calculations can be reformulated as follows:

Proposition 1.7. *A sub vector space $V \subset E_1 \oplus \cdots \oplus E_d$ is S -stable if and only if for any permutation $\tau \in \mathfrak{S}_d$, we have*

$$\dim(V)(x_{\tau(1)} + \cdots + x_{\tau(i)}) < \dim(V \cap (E_{\tau(1)} \oplus \cdots \oplus E_{\tau(i)})), \quad i = 1, \dots, d-1.$$

The same result holds for the semi-stable points with “ $<$ ” replaced by “ \leq ”.

1.2.2. *Comparison of two notions of stability.* First of all, we embed \mathcal{X}_n as a closed subvariety of some grassmannian.

Lemma 1.8. *The algebraic variety \mathcal{X}_n is a Springer fiber.*

Proof. For $L \in \mathcal{X}_n$, we have automatically $L \subset \epsilon^{(1-d)n}L_0$. Let Gr_{nd,nd^2} be the grassmannian of sub vector spaces of dimension nd in k^{nd^2} . We embed \mathcal{X}_n in Gr_{nd,nd^2} by the injective morphism $\varrho_n : \mathcal{X}_n \rightarrow \text{Gr}_{nd,nd^2}$ defined by

$$\varrho_n(L) = L/\epsilon^n L_0 \subset \epsilon^{(1-d)n}L_0/\epsilon^n L_0.$$

The image of ϱ_n is the Springer fiber

$$Y_n = \{V \in \text{Gr}_{nd,nd^2} \mid NV \subset V\},$$

where $N \in \text{End}(\epsilon^{(1-d)n}L_0/\epsilon^n L_0)$ is the endomorphism defined by the multiplication by ϵ . \square

For the reason of dimension, we use another embedding $\varrho_n^\perp : \mathcal{X}_n \rightarrow \text{Gr}_{nd(d-1), nd^2}$. We define an inner product on the vector space $k^{nd^2} = \epsilon^{(1-d)n} L_0 / \epsilon^n L_0$ by linearly expanding the relation

$$(\epsilon^{(1-d)n+i} e_j, \epsilon^{n-1+i'} e_{j'}) = \delta_{i,i'} \delta_{j,j'}, \quad i, i' = 0, \dots, dn-1; j, j' = 1, \dots, d,$$

For a sub vector space $V \subset k^{nd^2}$, let V^\perp be the orthogonal complement of V with respect to this inner product. It induces an isomorphism $^\perp : \text{Gr}_{nd, nd^2} \rightarrow \text{Gr}_{nd(d-1), nd^2}$. Let $\varrho_n^\perp(L) = \varrho_n(L)^\perp$, $\forall L \in \mathcal{X}_n$, and let Y_n denote again the image of \mathcal{X}_n in $\text{Gr}_{nd(d-1), nd^2}$ under ϱ_n^\perp .

Proof of proposition 1.4. Since the quotient \mathcal{X}^ξ/T doesn't depend on the choice of ξ , we can suppose that $\sum_{i=1}^d \xi_i = 0$, $\xi_i \in \mathbf{Q}$ is positive and small enough for $i = 1, \dots, d-1$.

Let $E_i = \mathfrak{p}^{(1-d)n} e_i / \mathfrak{p}^n e_i$, let $V = \varrho_n^\perp(L) \in Y_n \subset \text{Gr}_{nd(d-1), nd^2}$. For $\tau \in \mathfrak{S}_d$, we have the equality

$$(9) \quad \dim(V \cap (E_{\tau(1)} \oplus \dots \oplus E_{\tau(i)})) = \text{ind}(L \cap (Fe_{\tau(1)} \oplus \dots \oplus Fe_{\tau(i)})) + n(d-1)i.$$

Let $x_i = \frac{\xi_i + n(d-1)}{nd(d-1)}$. The hypothesis on ξ implies that $x_i \in \mathbf{Q}$, $x_i > 0$ and $\sum_{i=1}^d x_i = 1$. Take $r_i, s_i \in \mathbf{N}$ such that

$$x_d = \frac{1}{1 + \sum_{i=1}^{d-1} s_i/r_i}, \quad x_i = \frac{s_i/r_i}{1 + \sum_{i=1}^{d-1} s_i/r_i}, \quad i = 1, \dots, d-1.$$

Let $S_n \subset T$ be the torus defined in (1) for the above r_i, s_i . The equality (9) implies that

$$\xi_{\tau(1)} + \dots + \xi_{\tau(i)} < \text{ind}(L \cap (Fe_{\tau(1)} \oplus \dots \oplus Fe_{\tau(i)}))$$

if and only if

$$(10) \quad \dim(V)(x_{\tau(1)} + \dots + x_{\tau(i)}) < \dim(V \cap (E_{\tau(1)} \oplus \dots \oplus E_{\tau(i)})).$$

Combining the proposition 1.3 and the proposition 1.7, we get

$$L \in \mathcal{X}_n^\xi \iff V \in Y_n^{st}.$$

Since ξ is supposed to be generic and ξ_i are positive and small enough for $i = 1, \dots, d-1$, the “<” in the inequality (10) is the same as “ \leq ”. That is to say that $Y_n^{ss} = Y_n^{st}$ and so the quotient $Y_n^{ss}/S_n = Y_n^{st}/S_n$ is a projective k -variety by the geometric invariant theory of Mumford. The following lemma shows that $\mathcal{X}_n^\xi/T = \mathcal{X}_n^\xi/S_n \cong Y_n^{ss}/S_n$, since T/Z_G acts freely on \mathcal{X}_n^ξ . So \mathcal{X}_n^ξ/T is a projective k -variety. \square

Lemma 1.9. *The morphism $S_n \rightarrow T/Z_G$ is an isogeny.*

Proof. It is equivalent to show that the induced morphism of character groups $X^*(T/Z_G) \rightarrow X^*(S_n)$ has non zero determinant. By a direct calculation, this determinant is

$$(1 + \sum_{i=1}^{d-1} s_i/r_i) \prod_{i=1}^{d-1} r_i,$$

which is non zero by the choice of r_i, s_i . □

1.3. The groups Sp_{2d} and SO_{2d} . Let $(k^{2d}, \langle, \rangle)$ be the standard symplectic vector space over k such that $\langle e_i, e_{2d+1-i} \rangle = \delta_{i,j}$, $i, j = 1, \dots, d$. Let Sp_{2d} be the symplectic group associated to it, let T be the maximal torus of Sp_{2d} consisting of the diagonal matrices. Let $(F^{2d}, \langle, \rangle)$ be the scalar extension of $(k^{2d}, \langle, \rangle)$ to F . For a lattice L in F^{2d} , let

$$L^\vee = \{x \in F^{2d} \mid \langle x, L \rangle \subset \mathcal{O}\}.$$

The affine grassmannian associated to Sp_{2d} classifies the lattices L in F^{2d} such that $L = L^\vee$. Let

$$\mathcal{X}_n = \{L \in \mathcal{X} \mid \epsilon^n L_0 \subset L \subset \epsilon^{-n} L_0\}.$$

It is a T -invariant projective k -variety and we have $\mathcal{X} = \lim_{n \rightarrow +\infty} \mathcal{X}_n$. Let $\rho_n : \mathcal{X}_n \rightarrow \mathrm{Gr}_{2nd, 4nd}$ be the injective T -equivariant morphism defined by

$$\rho_n(L) = L / \epsilon^n L_0 \subset \epsilon^{-n} L_0 / \epsilon^n L_0.$$

Let Y_n be its image, it is isomorphic to \mathcal{X}_n . Let $\iota : \mathrm{Gr}_{2nd, 4nd} \rightarrow \mathbf{P}^N$, $N = \binom{4nd}{2nd} - 1$, be the Plücker embedding. Let $\mathcal{L} = (\iota \circ \rho_n)^* \mathcal{O}_{\mathbf{P}^N}(1)$, it is an ample T -equivariant line bundle on Y_n .

Let GSp_{2d} be the reductive group over k such that for any k -algebra R ,

$$\mathrm{GSp}_{2d}(R) = \{g \in \mathrm{GL}_{2d}(R) \mid \langle gv, gv' \rangle = \lambda(g) \langle v, v' \rangle, \lambda(g) \in R^\times, \forall v, v' \in R^{2d}\}.$$

We have an exact sequence

$$0 \rightarrow \mathrm{Sp}_{2d} \rightarrow \mathrm{GSp}_{2d} \xrightarrow{\lambda} \mathbb{G}_m \rightarrow 0,$$

from which it follows that \mathcal{X} is the neutral connected component of $\mathcal{X}^{\mathrm{GSp}_{2d}}$. Let

$$\tilde{T} = \left\{ \begin{bmatrix} tt_1 & & & & \\ & \ddots & & & \\ & & tt_d & & \\ & & & t_d^{-1} & \\ & & & & \ddots \\ & & & & & t_1^{-1} \end{bmatrix} ; t, t_i \in k^\times \right\}.$$

It is a maximal torus of GSp_{2d} . Let \mathbb{G}_m be the center of GSp_{2d} , then \tilde{T}/\mathbb{G}_m acts freely on \mathcal{X} and we have

$$\mathcal{X}^\xi / \tilde{T} = \mathcal{X}^\xi / T.$$

Given a generic element $\xi = (\xi_1, \dots, \xi_d, -\xi_d, \dots, -\xi_1) \in \mathfrak{t}$ such that $\xi_i \in \mathbf{Q}$, we can find $r_i, s_i \in \mathbf{Z}$, $i = 1, \dots, d$, such that

$$\xi_i = nd \frac{s_i / r_i}{2 + \sum_{i=1}^d s_i / r_i}, \quad i = 1, \dots, d$$

Consider the sub torus S_n of \tilde{T} defined by

$$S_n = \left\{ \begin{bmatrix} t_1^{s_1} & \cdots & t_d^{s_d} t_1^{r_1} & & & \\ & \ddots & & \ddots & & \\ & & t_1^{s_1} & \cdots & t_d^{s_d} t_d^{r_d} & \\ & & & & t_d^{-r_d} & \\ & & & & & \ddots \\ & & & & & & t_1^{-r_1} \end{bmatrix} ; t_i \in k^\times \right\}.$$

Lemma 1.10. *The morphism $S_n \rightarrow \tilde{T}/\mathbb{G}_m$ is an isogeny.*

Proof. As before, we need to calculate the determinant of the morphism of character groups $X^*(\tilde{T}/\mathbb{G}_m) \rightarrow X^*(S_n)$, it is

$$\left(2 + \sum_{i=1}^d s_i/r_i\right) \prod_{i=1}^d r_i,$$

which is non zero by the definition of r_i, s_i . \square

As we have done for GL_d , we can calculate the S_n -(semi)-stable points Y_n^{st} (resp. Y_n^{ss}) on Y_n with respect to the polarization given by the line bundle \mathcal{L} , and obtain the following comparison result, which implies the proposition 1.2.

Proposition 1.11. *Under the above setting, a lattice $L \in \mathcal{X}_n$ is ξ -stable if and only if $\rho_n(L) \in Y_n^{ss} = Y_n^{st}$. In particular, $\mathcal{X}_n^\xi/T = \mathcal{X}_n^\xi/(\tilde{T}/\mathbb{G}_m) \cong Y_n^{ss}/S_n$ is a projective k -variety.*

For the group SO_{2d} , the strategy is totally the same, unless we need to use the standard quadratic space $(k^{2d}, \langle, \rangle)$ over k such that $\langle e_i, e_{2d+1-i} \rangle = \delta_{i,j}$, $i, j = 1, \dots, 2d$, instead of the standard symplectic vector space.

1.4. The group SO_{2d+1} . Let $(k^{2d+1}, \langle, \rangle)$ be the standard quadratic space over k such that $\langle e_i, e_{2d+2-i} \rangle = \delta_{i,j}$, $i, j = 1, \dots, 2d+1$. Let SO_{2d+1} be the orthogonal group associated to it, and let T be the maximal torus of SO_{2d+1} consisting of the diagonal matrices. Let $(F^{2d+1}, \langle, \rangle)$ be the scalar extension of $(k^{2d+1}, \langle, \rangle)$ to F . For a lattice L in F^{2d+1} , let

$$L^\vee = \{x \in F^{2d+1} \mid \langle x, L \rangle \subset \mathcal{O}\}.$$

The affine grassmannian associated to SO_{2d+1} classifies the lattices L in F^{2d+1} such that $L = L^\vee$. Let

$$\mathcal{X}_n = \{L \in \mathcal{X} \mid \epsilon^n L_0 \subset L \subset \epsilon^{-n} L_0\}.$$

It is a T -invariant projective k -variety and

$$\mathcal{X} = \lim_{n \rightarrow +\infty} \mathcal{X}_n.$$

Let $\rho_n : \mathcal{X}_n \rightarrow \mathrm{Gr}_{n(2d+1), 2n(2d+1)}$ be the injective T -equivariant morphism defined by

$$\rho_n(L) = L/\epsilon^n L_0 \subset \epsilon^{-n} L_0/\epsilon^n L_0.$$

Let Y_n be its image, it is isomorphic to \mathcal{X}_n . Let \mathcal{L} again be the T -equivariant line bundle on Y_n induced by the Plücker embedding of $\mathrm{Gr}_{n(2d+1), 2n(2d+1)}$.

Let GO_{2d+1} be the reductive group over k such that for any k -algebra R ,

$$\mathrm{GO}_{2d+1}(R) = \{g \in \mathrm{GL}_{2d+1}(R) \mid \langle gv, gv' \rangle = \lambda(g) \langle v, v' \rangle, \lambda(g) \in R^\times, \forall v, v' \in R^{2d+1}\}.$$

We have an exact sequence

$$0 \rightarrow \mathrm{SO}_{2d+1} \rightarrow \mathrm{GO}_{2d+1} \xrightarrow{\lambda} \mathbb{G}_m \rightarrow 0,$$

from which it follows that \mathcal{X} is the neutral connected component of $\mathcal{X}^{\mathrm{GO}_{2d+1}}$. Let

$$\tilde{T} = \left\{ \begin{bmatrix} t^2 t_1 & & & & & \\ & \ddots & & & & \\ & & t^2 t_d & & & \\ & & & t & & \\ & & & & t_d^{-1} & \\ & & & & & \ddots \\ & & & & & & t_1^{-1} \end{bmatrix} ; t, t_i \in k^\times \right\}.$$

It is a maximal torus of GO_{2d+1} . Let \mathbb{G}_m be the center of GO_{2d+1} , then \tilde{T}/\mathbb{G}_m acts freely on \mathcal{X} and we have

$$\mathcal{X}^\xi / \tilde{T} = \mathcal{X}^\xi / T.$$

Given a generic element $\xi = (\xi_1, \dots, \xi_d, 0, -\xi_d, \dots, -\xi_1) \in \mathfrak{t}$ such that $\xi_i \in \mathbf{Q}$, we can find $r_i, s_i \in \mathbf{Z}$, $i = 1, \dots, d$, such that

$$\xi_i = n(d + \frac{1}{2}) \frac{s_i/r_i}{1 + \sum_{i=1}^d s_i/r_i}, \quad i = 1, \dots, d$$

Consider the sub torus S_n of \tilde{T} defined by

$$S_n = \left\{ \begin{bmatrix} t_1^{2s_1} \dots t_d^{2s_d} t_1^{r_1} & & & & & \\ & \ddots & & & & \\ & & t_1^{2s_1} \dots t_d^{2s_d} t_d^{r_d} & & & \\ & & & t_1^{s_1} \dots t_d^{s_d} & & \\ & & & & t_d^{-r_d} & \\ & & & & & \ddots \\ & & & & & & t_1^{-r_1} \end{bmatrix} ; t_i \in k^\times \right\}.$$

Lemma 1.12. *The morphism $S_n \rightarrow \tilde{T}/\mathbb{G}_m$ is an isogeny.*

Proof. As before, we need to calculate the determinant of the morphism of character groups $X^*(\tilde{T}/\mathbb{G}_m) \rightarrow X^*(S_n)$, it is

$$(1 + \sum_{i=1}^d s_i/r_i) \prod_{i=1}^d r_i,$$

which is non zero by the definition of r_i, s_i . \square

As before, we can calculate the S_n -(semi)-stable points Y_n^{st} (resp. Y_n^{ss}) on Y_n with respect to the polarization given by the line bundle \mathcal{L} , and obtain the following comparison result. It implies the proposition 1.2.

Proposition 1.13. *Under the above setting, a lattice $L \in \mathcal{X}_n$ is ξ -stable if and only if $\rho_n(L) \in Y_n^{ss} = Y_n^{st}$. In particular, $\mathcal{X}_n^\xi/T = \mathcal{X}_n^\xi/(\tilde{T}/\mathbb{G}_m) \cong Y_n^{ss}/S_n$ is a projective k -variety.*

2. REDUCTION OF ARTHUR-KOTTWITZ

We will introduce an analogue of the Harder-Narasimhan reduction on the affine grassmannian, which we will name *the reduction of Arthur-Kottwitz*.

For $P \in \mathcal{F}(T)$, let $P = MN$ be the standard Levi factorization. Let $\Phi_P(G, M)$ be the image of $\Phi(N, T)$ in $(\mathfrak{a}_M^G)^*$. For any point $a \in \mathfrak{a}_M^G$, we define a cone in \mathfrak{a}_M^G ,

$$D_P(a) = \{y \in \mathfrak{a}_M^G \mid \alpha(y - a) \geq 0, \forall \alpha \in \Phi_P(G, M)\}.$$

Definition 2.1. For any geometric point $x \in \mathcal{X}$, we define a semi-cylinder $C_P(x)$ in \mathfrak{a}_T^G by

$$C_P(x) = \pi^{M,-1}(\text{Ec}^M(f_P(x))) \cap \pi_M^{-1}(D_P(H_P(x))).$$

By definition, we get a partition

$$\mathfrak{a}_T^G = \text{Ec}(x) \cup \bigcup_{P \in \mathcal{F}(T)} C_P(x),$$

such that the interior of any two members doesn't intersect. The figure 1 gives an idea of this partition for GL_3 .

So for any $x \notin \mathcal{X}^\xi$, there exists a unique parabolic subgroup $P \in \mathcal{F}(T)$ such that $\xi \in C_P(x)$ since ξ is generic. In this case, $f_P(x) \in \mathcal{X}^M$ is ξ^M -stable, where $\xi^M = \pi^M(\xi) \in \mathfrak{a}_T^M$. We define

$$S_P = \{x \in \mathcal{X} \mid \xi \in C_P(x)\}.$$

Lemma 2.1. *We have a stratification of the affine grassmannian*

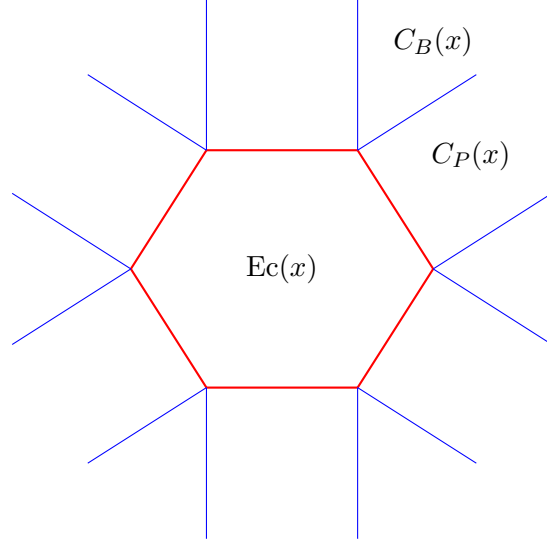
$$\mathcal{X} = \mathcal{X}^\xi \sqcup \bigsqcup_{P \in \mathcal{F}(T), P \neq G} S_P.$$

For $P \in \mathcal{P}(M)$, let P^- be the parabolic subgroup opposite to P with respect to M . Let $\Lambda_{M,P} = D_{P^-}(\xi_M) \cap \Lambda_M$, where $\xi_M = \pi_M(\xi) \in \mathfrak{a}_M^G$. We have the disjoint partition

$$\Lambda_M = \bigsqcup_{P \in \mathcal{P}(M)} \Lambda_{M,P}.$$

Let

$$\mathcal{X}_P^M = \text{ind}^{M,-1}(\Lambda_{M,P}), \quad \mathcal{X}_P^{M,\xi^M} = \mathcal{X}^{M,\xi^M} \cap \mathcal{X}_P^M.$$

FIGURE 1. $\text{Ec}(x)$ and $C_P(x)$ for GL_3 .

We remind that \mathcal{X}^M is embedded naturally in \mathcal{X} by the inclusion $M(F) \rightarrow G(F)$. In this way, \mathcal{X}_P^{M, ξ^M} is an embedding naturally in S_P .

Proposition 2.2. *We have $S_P = N(F)\mathcal{X}_P^{M, \xi^M}$.*

Proof. By definition, we have $f_P(S_P) \subset \mathcal{X}_P^{M, \xi^M}$, so $S_P \subset N(F)\mathcal{X}_P^{M, \xi^M}$.

For the inverse inclusion, let $x \in \mathcal{X}_P^{M, \xi^M} \subset S_P$, $u \in N(F)$, then $f_P(ux) = f_P(x)$ and so $H_P(ux) = H_P(x)$. They imply that $C_P(ux) = C_P(x)$, so $ux \in S_P$. \square

The lemma 2.1 and the proposition 2.2 enable us to reduce the affine grassmannian \mathcal{X} into the ξ^M -stable parts of \mathcal{X}^M , $M \in \mathcal{L}(T)$. This process is called *the reduction of Arthur-Kottwitz*.

3. POINCARÉ SERIES OF \mathcal{X}^ξ/T FOR SL_d

Let $k = \overline{\mathbf{F}}_p$, let l be a prime number different from p . Let $G = \text{SL}_d$, let T be the maximal torus of G consisting of the diagonal matrices, let B be the Borel subgroup of G consisting of the upper triangular matrices. Let $X_*^+(T)$ be the cone of dominant cocharacters μ of T with respect to B .

3.1. Poincaré series of the affine grassmannian. Let V be a separated scheme of finite type over k , we use the notation:

$$H_i(V) = (H^i(V, \overline{\mathbf{Q}}_l))^*, \quad H_{i,c}(V) = (H_c^i(V, \overline{\mathbf{Q}}_l))^*.$$

Its Poincaré polynomial is defined to be

$$P_V(t) = \sum_{i=0}^{2 \dim(V)} \dim(H_i(V)) t^i.$$

For an ind- k -scheme $\mathcal{V} = \lim_{n \rightarrow +\infty} V_n$, we define

$$H_i(\mathcal{V}) = \lim_{n \rightarrow \infty} H_i(V_n), \quad H_{i,c}(\mathcal{V}) = \lim_{n \rightarrow \infty} H_{i,c}(V_n).$$

If $\dim(H_i(\mathcal{V})) < +\infty$ for all $i \in \mathbf{N}$, we define the Poincaré series of \mathcal{V} to be

$$P_{\mathcal{V}}(t) = \sum_{i=0}^{\infty} \dim(H_i(\mathcal{V})) t^i.$$

Proposition 3.1 (Bott). *The Poincaré series of the affine grassmannian is*

$$P_{\mathcal{X}}(t) = \prod_{i=1}^{d-1} (1 - t^{2i})^{-1}.$$

The reader can find in [B] a topological proof, and in [IM] a combinatorial proof.

3.2. \mathcal{X}^{ξ}/T is homologically smooth. The aim of this section is to prove the following result:

Lemma 3.2. *For any $n \in \mathbf{N}$, the algebraic variety \mathcal{X}_n^{ξ}/T is homologically smooth. In particular, it satisfies the Poincaré duality.*

The proof is based on an observation of Lusztig in [L], later generalized by Mirkovic and Vybornov in [MVy], which says that the affine grassmannian has the same singularity as the nilpotent cone.

For $\mu \in X_*^+(T)$, let $\text{Sch}(\mu) = \overline{K\epsilon^{\mu}K/K}$. We have the stratification in K -orbits

$$\text{Sch}(\mu) = \bigcup_{\substack{\lambda \in X_*^+(T) \\ \lambda \prec \mu}} K\epsilon^{\lambda}K/K,$$

where $\lambda \prec \mu$ means:

$$\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i, \quad i = 1, \dots, d.$$

So $\text{Sch}(\mu)$ is equisingular along $K\epsilon^{\lambda}K/K$, hence the local singularity of $\text{Sch}(\mu)$ along $K\epsilon^{\lambda}K/K$ is the same as that of a transversal slice to $K\epsilon^{\lambda}K/K$.

Let $L^{<0}G = \{g \in \text{GL}_d(k[[\epsilon^{-1}]]) \mid g \equiv 1 \pmod{\epsilon^{-1}}\}$.

Lemma 3.3. *For $\lambda \in X_*^+(T)$, the orbit $L^{<0}G \cdot \epsilon^{\lambda}$ is a transversal slice to $K\epsilon^{\lambda}K/K$ passing the point ϵ^{λ} in the affine grassmannian \mathcal{X} .*

The reader can consult [BL] for a proof.

For $N \in \mathbf{N}$, let \mathcal{N} be the nilpotent cone of \mathfrak{gl}_N .

Theorem 3.4 (Borho-Macpherson). *The nilpotent cone \mathcal{N} is homologically smooth.*

The reader can consult [BM] for a proof.

Let $p(N)$ be the set of partitions of N . By a partition of N , we mean a tuple of numbers $(a_1, \dots, a_n) \in \mathbf{N}$, $a_1 \geq \dots \geq a_n \geq 1$ such that $a_1 + \dots + a_n = N$. For $\lambda \in p(N)$, let u_λ be the Jordan matrix of type λ and let \mathcal{O}_λ be the orbit of u_λ under the conjugation action of GL_N . For $\mu \in p(N)$, we have the stratification in GL_N -orbits

$$\overline{\mathcal{O}_\mu} = \bigsqcup_{\substack{\lambda \in p(N) \\ \lambda \prec \mu}} \mathcal{O}_\lambda,$$

where $\lambda \prec \mu$ means that for any i , we have

$$\lambda_1 + \dots + \lambda_i \leq \mu_1 + \dots + \mu_i.$$

Thus $\overline{\mathcal{O}_\mu}$ is equisingular along \mathcal{O}_λ , and so the local singularity of $\overline{\mathcal{O}_\mu}$ along \mathcal{O}_λ is the same as that of a transversal slice to \mathcal{O}_λ in $\overline{\mathcal{O}_\mu}$.

In [MVy], Mirkovic and Vybournov construct a transversal slice to \mathcal{O}_λ in $\overline{\mathcal{O}_\mu}$. We review briefly their construction. Let $\lambda = (\lambda_1, \dots, \lambda_r)$, let $\{e_{i,j}, i = 1, \dots, \lambda_j; j = 1, \dots, r\}$ be the standard basis of k^N such that

$$u_\lambda e_{1,j} = 0, \quad \text{et} \quad u_\lambda e_{i,j} = e_{i-1,j}, \quad i = 2, \dots, \lambda_j; \quad j = 1, \dots, r.$$

Let $V_j = \bigoplus_{i=1}^{\lambda_j} k e_{i,j}$, $j = 1, \dots, r$. With respect to this basis, we identify

$$\mathfrak{gl}_N = \bigoplus_{i,j=1, \dots, r} \mathrm{Hom}(V_i, V_j).$$

Let $u_i = u_\lambda|_{V_i}$, and

$$C = \bigoplus_{i,j=1, \dots, r} \mathrm{Hom}(\ker(u_i^{\lambda_j}), \ker(u_j^t)),$$

where u_j^t is the transposition of u_j .

Lemma 3.5 (Mirkovic-Vybournov). *The sub-variety $(u_\lambda + C) \cap \overline{\mathcal{O}_\mu}$ is a transversal slice to \mathcal{O}_λ passing through u_λ in $\overline{\mathcal{O}_\mu}$.*

Take any $m \in \mathbf{N}$, $m \geq \lambda_1$, let $\mathbf{m} - \lambda = (m - \lambda_d, \dots, m - \lambda_1)$, it is a partition of dm .

Theorem 3.6 (Mirkovic-Vybournov). *For $\lambda, \mu \in X_*^+(T)$, $\lambda \prec \mu$, there exists an isomorphism*

$$(u_{\mathbf{m}-\lambda} + C) \cap \overline{\mathcal{O}_{\mathbf{m}-\mu}} \cong (L^{<0}G \cdot \epsilon^\lambda) \cap \mathrm{Sch}(\mu).$$

So the singularity of $\mathrm{Sch}(\mu)$ along $K\epsilon^\lambda K/K$ is the same as that of $\overline{\mathcal{O}_{\mathbf{m}-\mu}}$ along $\mathcal{O}_{\mathbf{m}-\lambda}$. In particular, we have

Corollary 3.7. *For $n \in \mathbf{N}$, the algebraic variety \mathcal{X}_n is homologically smooth.*

Proof. Take $\nu = (n, \dots, n, -(d-1)n) \in X_*^+(T)$, we have $\mathcal{X}_n = \mathrm{Sch}(\nu)$. Now that $\overline{\mathcal{O}_{\mathbf{n}-\nu}}$ is the nilpotent cone in \mathfrak{gl}_{dn} , the corollary follows from theorem 3.4. \square

Proof of the lemma 3.2. Since \mathcal{X}_n^ξ is open in \mathcal{X}_n , \mathcal{X}_n^ξ is homologically smooth. It is a T/\mathbb{G}_m -torsor over \mathcal{X}_n^ξ/T since the action of T/\mathbb{G}_m is free. In particular, the natural projection $\mathcal{X}_n^\xi \rightarrow \mathcal{X}_n^\xi/T$ is a surjective smooth morphism. According to [G], corollary 17.16.3, locally it admits an étale section. So the torsor is locally trivial for the étale topology. This implies that \mathcal{X}_n^ξ/T is homologically smooth since \mathcal{X}_n^ξ is. \square

3.3. Calculation of the Poincaré series.

Lemma 3.8. *Let $\xi \in \mathfrak{a}_T^G$ be such that $0 < \alpha(\xi) < 1$, $\forall \alpha \in \Phi_B(G, T)$. We have $\dim(\mathcal{X}_n) = nd(d-1)$, and the closed sub-variety $\mathcal{X}_n \setminus \mathcal{X}_n^\xi$ have dimension at most $n(d-1)^2$, i.e. its codimension is at least $(d-1)n$ in \mathcal{X}_n .*

Proof. Let $\nu = (n, \dots, n, (1-d)n) \in X_*(T)$, then $\dim(\mathcal{X}_n) = \dim(Ie^\nu K/K) = nd(d-1)$ since $\mathcal{X}_n = \overline{Ie^\nu K/K}$. Here I is the standard Iwahori subgroup of $G(F)$, i.e. it is the inverse image of B under the reduction morphism $G(\mathcal{O}) \rightarrow G(k)$.

The dimension of the closed sub-variety $\mathcal{X}_n \setminus \mathcal{X}_n^\xi$ is

$$\max\{\dim(S_P \cap \mathcal{X}_n), P \in \mathcal{F}(T), P \neq G\}.$$

An easy induction reduces the situation to the case where P is a standard maximal parabolic subgroup, here standard means that $B \subset P$. Suppose that P is of type $(r, d-r)$, $1 \leq r \leq d-1$, i.e. its Levi factor is $M = \mathrm{SL}(k^r \oplus k^{d-r})$.

We identify Λ_M with $\{(\lambda, -\lambda) \mid \lambda \in \mathbf{Z}\}$. For $(\lambda, -\lambda) \in \Lambda_{M,P}$, we have $\lambda \leq 0$. Let $X^\lambda = H_P^{-1}((\lambda, -\lambda)) \cap \mathcal{X}_n$, then the $S_P^\lambda := S_P \cap X^\lambda$ are the connected components of $S_P \cap \mathcal{X}_n$. So it is enough to bound the dimension of X^λ .

Lemma 3.9. *We have the affine paving*

$$\mathcal{X}_n = \bigcup_{\epsilon^\mu \in \mathcal{X}_n^T} \mathcal{X}_n \cap B\epsilon^\mu K/K,$$

where

$$\mathcal{X}_n \cap B\epsilon^\mu K/K = \begin{bmatrix} 1 & \cdots & \mathbf{p}^{a_{i,j}} \\ & \ddots & \vdots \\ & & 1 \end{bmatrix}^{-1} \epsilon^\mu K/K$$

with $a_{i,j} = \mu_i - n$, is isomorphic to an affine space of dimension

$$\sum_{i=2}^d (i-1)(n - \mu_i).$$

For the proof, the reader can refer to [C], Prop. 2.2 and Cor. 2.5. For $\epsilon^\mu \in \mathcal{X}_n^T$, let $C(\mu) = \mathcal{X}_n \cap B\epsilon^\mu K/K$, it is of dimension

$$\sum_{i=2}^d (i-1)(n - \mu_i) = \sum_{i=2}^r (i-1)(n - \mu_i) + \sum_{i=r+1}^d (i-1)(n - \mu_i).$$

Since $B \subset P$, we have

$$X^\lambda = \bigsqcup_{\epsilon^\mu \in (X^\lambda)^T} C(\mu).$$

So the question is to bound the dimension of $C(\mu)$ under the condition that $\mu_i \leq n$ and that

$$\sum_{i=1}^r \mu_i = \lambda \leq 0, \quad \sum_{i=r+1}^d \mu_i = -\lambda \geq 0.$$

It takes the maximal value when

$$\begin{aligned} \mu_1 &= \cdots = \mu_{r-1} = n, \mu_r = \lambda - (r-1)n; \\ \mu_{r+1} &= \cdots = \mu_{d-1} = n, \mu_d = -\lambda - (d-r-1)n. \end{aligned}$$

So we have

$$\begin{aligned} \dim(X^\lambda) &\leq (r-1)(n + (r-1)n - \lambda) + (d-1)(n + (d-r-1)n + \lambda) \\ &= (r-1)rn + (d-1)(d-r)n + (d-r)\lambda \leq (d-1)^2n, \end{aligned}$$

and then

$$\text{Codim}(\mathcal{X}_n \setminus \mathcal{X}_n^\xi) \geq nd(d-1) - (d-1)^2n = (d-1)n.$$

□

Remark 3.1. We can also use the dimension formula in theorem 3.2 of [MVi] to obtain the same estimation.

For $n \in \mathbf{N}$, let τ_n be the truncation operator on $k[[t]]$ defined by

$$\tau_n \left(\sum_{i=i_0}^{+\infty} a_i t^i \right) = \sum_{i=i_0}^n a_i t^i.$$

Theorem 3.10. *The Poincaré series of \mathcal{X}^ξ/T is*

$$P_{\mathcal{X}^\xi/T}(t) = \frac{1}{(1-t^2)^{d-1}} \prod_{i=1}^{d-1} (1-t^{2i})^{-1}.$$

Further more, we have

$$H_{2i+1}(\mathcal{X}^\xi/T) = 0,$$

and the Frobenius acts on $H_{2i}(\mathcal{X}^\xi/T)$ by q^{-i} , $\forall i \geq 0$.

Proof. We have the exact sequence

$$\cdots \rightarrow H_i(\mathcal{X}_n \setminus \mathcal{X}_n^\xi) \rightarrow H_i(\mathcal{X}_n) \rightarrow H_{i,c}(\mathcal{X}_n^\xi) \rightarrow H_{i-1}(\mathcal{X}_n \setminus \mathcal{X}_n^\xi) \rightarrow \cdots$$

By lemme 3.8,

$$\dim(\mathcal{X}_n \setminus \mathcal{X}_n^\xi) \leq n(d-1)^2,$$

so

$$H_i(\mathcal{X}_n \setminus \mathcal{X}_n^\xi) = 0, \quad i \geq 2n(d-1)^2 + 1,$$

and

$$H_{i,c}(\mathcal{X}_n^\xi) = H_i(\mathcal{X}_n), \quad i \geq 2n(d-1)^2 + 2.$$

Since \mathcal{X}_n and \mathcal{X}_n^ξ are homologically smooth, they satisfy the Poincaré duality, which implies that

$$(11) \quad H_i(\mathcal{X}_n^\xi) = H_i(\mathcal{X}_n), \quad 0 \leq i \leq 2n(d-1) - 2,$$

since $\dim(\mathcal{X}_n) = \dim(\mathcal{X}_n^\xi) = nd(d-1)$.

Because T/\mathbb{G}_m acts freely on \mathcal{X}_n^ξ , we have

$$(12) \quad \begin{aligned} H_i(\mathcal{X}_n^\xi/T) &= H_{i,T/\mathbb{G}_m}(\mathcal{X}_n^\xi) \\ &= \bigoplus_{i_1+i_2=i} H_{i_1}(\mathcal{X}_n^\xi) \otimes H_{i_2}(B(T/\mathbb{G}_m)), \end{aligned}$$

where $B(T/\mathbb{G}_m)$ is the classifying space of T/\mathbb{G}_m -torsors.

Combining the equalities (11) and (12), we get

$$(13) \quad \tau_{2(d-1)n-2}[P_{\mathcal{X}_n^\xi/T}(t)] = \tau_{2(d-1)n-2}[(1-t^2)^{1-d}P_{\mathcal{X}_n}(t)],$$

and

$$H_{2i+1}(\mathcal{X}_n^\xi/T) = 0, \quad 0 \leq i \leq (d-1)n - 2,$$

and the Frobenius acts on $H_{2i}(\mathcal{X}_n^\xi/T)$ by q^{-i} , $0 \leq i \leq (d-1)n - 1$, because it acts thus on $H_*(\mathcal{X}_n)$ and $H_*(B(T/\mathbb{G}_m))$.

Since

$$\lim_{n \rightarrow +\infty} \tau_{2(d-1)n}(P_{\mathcal{X}_n}(t)) = P_{\mathcal{X}}(t) = \prod_{i=1}^{d-1} (1-t^{2i})^{-1},$$

we can take limits of the two sides of (13), and get:

$$P_{\mathcal{X}^\xi/T}(t) = \frac{1}{(1-t^2)^{d-1}} \prod_{i=1}^{d-1} (1-t^{2i})^{-1}.$$

This implies that we can also take limits of the two sides of (11) and (12), and obtain

$$H_i(\mathcal{X}^\xi/T) = \bigoplus_{i_1+i_2=i} H_{i_1}(\mathcal{X}) \otimes H_{i_2}(B(T/\mathbb{G}_m)),$$

and the second part of the theorem follows. \square

Remark 3.2. The rotation torus \mathbb{G}_m acts on the quotient \mathcal{X}^ξ/T . It gives an affine paving of \mathcal{X}^ξ/T for SL_2 . But for SL_d , $d \geq 3$, the fixed points $(\mathcal{X}^\xi/T)^{\mathbb{G}_m}$ are not discrete.

REFERENCES

- [A] J. Arthur, *The characters of discrete series as orbital integrals*, Invent. math. **32**(1976), 205-261.
- [B] R. Bott, *An application of the Morse theory to the topology of Lie groups*, Bull. Soc. Math. France, 84 (1956), p.251-281.
- [BL] A. Beauville, Y. Laszlo, *Conformal blocks and generalized theta functions*, Commun. Math. Phys. 164, 385-419 (1994).
- [BM] W. Borho, R. Macpherson, *Partial resolutions of nilpotent varieties*, Analysis and topology on singular spaces II, III. 23-74, Astérisque, 101-102. Soc. Math. France, Paris, 1983
- [C] Zongbin Chen, *Pureté des fibres de Springer affines pour GL_4* , <http://arxiv.org/abs/1111.3352>
- [CL] P-H. Chaudouard, G. Laumon, *Le lemme fondamental pondéré I: constructions géométriques*, Compos. Math. 146 (2010), no. 6, 1416-1506.
- [G] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*. Publ. Math. IHES. 32, 1967, 5-361.
- [IM] N. Iwahori, H. Matsumoto, *On some Bruhat decomposition and the structure of the Hecke rings of p -adic Chevalley groups*, Publ. Math. IHES, 25 (1965), 5-48.
- [L] G. Lusztig, *Green polynomials and singularities of unipotent classes*, Adv. in Math. 42, (1981), 169-178.
- [M] D. Mumford, *Geometric invariant theory*, Springer-Verlag, Berlin-New York, 1965.
- [MVi] I. Mirkovic, K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. (2) 166 (2007), no.1, 95-143.
- [MVy] I. Mirkovic, M. Vybornov, *Quiver varieties and Beilinson-Drinfeld Grassmannian of type A*, <http://arxiv.org/abs/0712.4160>

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